C. Homotopy groups of spheres

we will use spectral sequences to compute homotopy groups of spheres, first we show

Th$^{\underline{m}}$ 9: $\boxed{\pi_4(S^3) \cong \mathbb{Z}/2}$

Remark: recall $\pi_3(S^2) \cong \mathbb{Z}$

so $\pi_{n+1}(S^n)$ not fixed

it will take awhile to prove Th$^{\underline{m}}$ 9, we start with a def$^{\underline{n}}$

if X is a CW complex then, then for each n

$\exists$ a sequence of fibrations

$$\begin{tikzcd} K(\pi_q(X), q) \arrow[r] & Y_q \arrow[d] \\ & Y_{q-1} \end{tikzcd} \qquad q = 1, \dots, n$$

and maps $X \xrightarrow{f_q} Y_q$

s.t. 1) $\pi_k(X) \xrightarrow{(f_q)_*} \pi_k(Y_q)$ is isomorphism $\forall\, k \leq q$

2) $\pi_k(Y_q) = 0 \quad \forall\, k > q$

3)
$$\begin{tikzcd}
K(\pi_n(X), n) \arrow[r] & Y_n & \\
& \vdots \arrow[d] & \\
K(\pi_3(X), 3) \arrow[r] & Y_3 \arrow[d, "p_3"] & \\
K(\pi_2(X), 2) \arrow[r] & Y_2 \arrow[d, "p_2"] & \\
K(\pi_1(X), 1) \arrow[r, equal] & Y_1 & X \arrow[l, "f_1"'] \arrow[ul, bend right, "f_2"'] \arrow[uul, bend right, "f_3"'] \arrow[uuuul, bend right, "f_n"']
\end{tikzcd}$$

commutes

this is called a Postnikov tower for X

and the Y_q are Postnikov approximations of X

lemma 10:

> for each n, every CW-complex has a Postnikov tower

Proof: given a CW-complex X, by Th$^{\underline{m}}$ I.27 we can attach cells of dimension $\geq n+2$ to kill the homotopy groups $\pi_k(X)$, $k>n$, without changing π_k for $k \leq n$

call resulting space Y_n

build Y_{n-1} by attaching cells of dim $\geq n+1$ to kill the homotopy groups π_k, $k>n-1$

continuing we get $X \subset Y_n \subset Y_{n-1} \subset \dots \subset Y_1$

and the inclusion $f_q: X \to Y_q$ induces an isomorphism on π_k, $k \leq q$

on page 22 of Section II we showed how to turn an inclusion into a fibration (upto homotopy) so we get fibrations

$$Y_q \longrightarrow Y_{q-1}$$

now the long exact sequence of a fibration (Cor II.7) gives

$$\pi_{k+1}(Y_q) \to \pi_{k+1}(Y_{q-1}) \to \pi_k(F) \to \pi_k(Y_q) \to \pi_k(Y_{q-1})$$

if $k \neq q$ or $q-1$, the outermost maps are isomorphisms so $\pi_k(F) = 0$

for $k = q$

$$0 \to \pi_q(F) \to \pi_q(Y_q) \xrightarrow{p_*} \pi_q(Y_{q-1})$$

$$\pi_q(F) = \ker p_* \underset{\text{by construction}}{=} \pi_q(X)$$

for $k = q-1$

$$\underset{\substack{\| \\ 0}}{\pi_q(Y_{q-1})} \to \pi_{q-1}(F) \to \pi_{q-1}(Y_q) \xrightarrow{\cong} \pi_{q-1}(Y_{q-1})$$

so F is a $K(\pi_q(X), q)$

<u>lemma 11</u>:

$$\boxed{\begin{array}{l} \pi_q(S^3) \cong H_{q+1}(Y_{q-1}) \quad \text{for } q > 3 \\ \text{where } Y_{q-1} \text{ is the } (q-1)^{\text{st}} \text{ term in a Postnikov tower for } S^3 \end{array}}$$

<u>Proof</u>: the Postnikov tower for S^3 has

$Y_1 \simeq Y_2 \simeq pt$ (since only nontrivial π_k is π_0)

$Y_3 = K(\pi_3(S^3), 3) = K(\mathbb{Z}, 3)$

$Y_q = S^3 \cup (q+2)\text{-cells} \cup (q+3)\text{-cells} \cup \ldots$

<u>note</u>: $H_q(Y_q) = H_{q+1}(Y_q) = 0$ for $q > 3$

since Y_q has no q or $q+1$ cells!

consider the fibration

$$\begin{array}{ccc} K(\pi_q(S^3), q) & \longrightarrow & Y_q \\ & & \downarrow \\ & & Y_{q-1} \end{array}$$

we compute the homology of Y_q using the Leray-Serre spectral sequence for this

we need to know

$$H_t(K(\pi_q(S^3),q)) \underset{\text{Hurewicz}}{=} \begin{cases} \mathbb{Z} & t=0 \\ 0 & t=1,\dots,q-1 \\ \pi_q(S^3) & t=q \\ ? & t>q \end{cases}$$

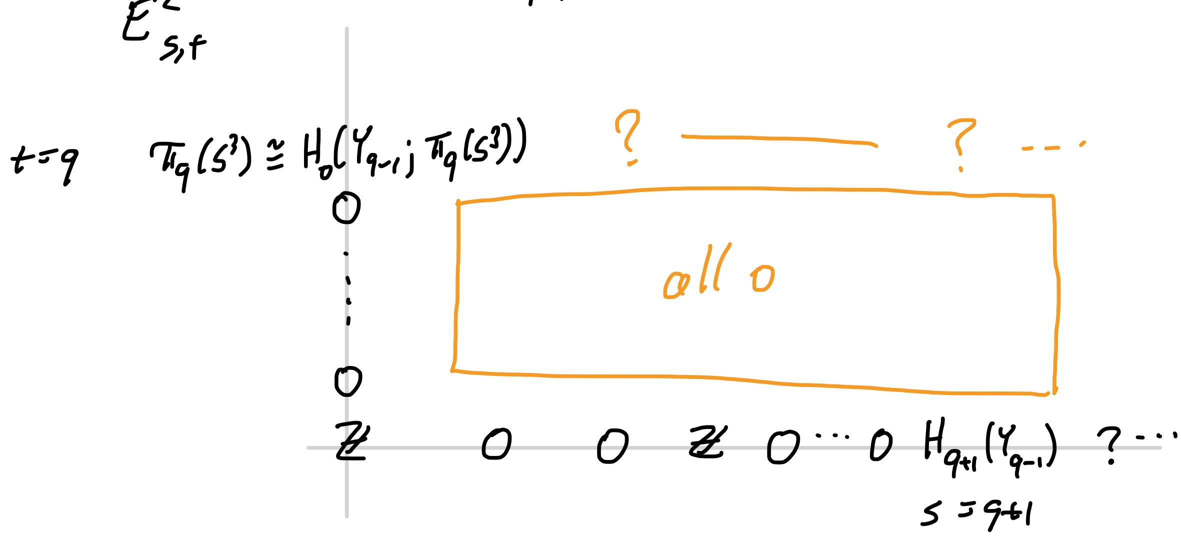
the E^2- term of spectral sequence is

$$E^2_{s,t} = H_s(Y_{q-1}; H_t(K(\pi_q(S^3)),q))$$

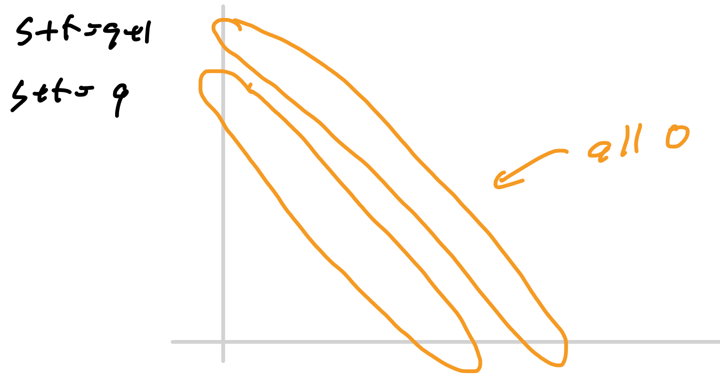
as above note

$$H_s(Y_{q-1}) = \begin{cases} \mathbb{Z} & s=0 \\ 0 & s=1 \\ 0 & s=2 \\ \mathbb{Z} & s=3 \\ 0 & s=4 \\ \vdots & \\ 0 & s=q \\ ? & \end{cases}$$

since $Y_{q-1} = S^3 \cup$ k-cells with $k \geq q+1$



We know $E^{\infty}_{s,t}$ is



note: all $d^k : E^k_{k,q-k+1} \longrightarrow E^k_{0,q}$

are zero except when $k = q+1$

$\therefore$ since $E^{\infty}_{0,q} = 0$ must have d^k onto

but if $\ker d^k \neq 0$ then $E^{\infty}_{q+1,0} = \ker d^k$

but must be 0

$\therefore$ d^k an isomorphism

$$\begin{array}{ccc} E^{q+1}_{q+1,0} & \longrightarrow & E^{q+1}_{0,q} \\ \wr\Vert & & \wr\Vert \\ H_{q+1}(Y_{q-1}) & & \pi_q(S^3) \end{array}$$

Cor 12:

$$\boxed{\pi_4(S^3) \cong H_5(K(\mathbb{Z},3))}$$

Proof: by last proof we have

$$\pi_4(S^3) = H_5(Y_3) = H_5(K(\pi_3(S^3),3)) = H_5(K(\mathbb{Z},3))$$

Thm 13:

$$\boxed{H_5(K(\mathbb{Z},3)) \cong \mathbb{Z}/2}$$

Proof of Th$^{\underline{m}}$ 9:

$$\text{Cor 12 and Th}^{\underline{m}}\,13 \Rightarrow \pi_4(S^3) \cong \mathbb{Z}/2$$

Proof of Th$^{\underline{m}}$ 13:

we start by computing some of the cohomology of $K(\mathbb{Z},3)$

recall from Cor II.9 for any X

$$\pi_{n-1}(\Omega X) \cong \pi_n(X)$$

so $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) = \mathbb{C}P^\infty$

indeed
$$\begin{array}{ccc} S^1 & \to & S^\infty \simeq * \\ & & \downarrow \\ & & \mathbb{C}P^\infty \end{array}$$
is a fibration

and $S^\infty \simeq pt$ so

$$\underbrace{\pi_n(S^\infty)}_{0} \to \pi_n(\mathbb{C}P^\infty) \xrightarrow{\cong} \pi_{n-1}(S^1) \to \underbrace{\pi_{n-1}(S^\infty)}_{0}$$

we have a fibration (by lemma II.5)

$$\begin{array}{ccc} K(\mathbb{Z},2) \simeq \Omega K(\mathbb{Z},3) & \to & PK(\mathbb{Z},3) \\ & & \downarrow \\ & & K(\mathbb{Z},3) \end{array}$$

and $PK(\mathbb{Z},3) \simeq pt$

the cohomology Leray-Serre spectral sequence has

$$E_2^{s,t} = H^s(K(\mathbb{Z},3), H^t(\mathbb{C}P^\infty))$$

recall $H^*\mathbb{C}P^\infty \cong \mathbb{Z}[a]$ degree $a = 2$

and $H_1(K(\mathbb{Z},3)) = H_2(K(\mathbb{Z},3)) = 0$ since $\pi_1 = \pi_2 = 0$ and Hurewicz

$\therefore H^1(K(\mathbb{Z},3)) = H^2(K(\mathbb{Z},3)) = 0$ by Universal Coeffic. Th$^{\underline{m}}$

so $E_2^{s,t}$ is

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathbb{Z}a^3 & 0 & 0 & \\
0 & 0 & 0 & \\
\mathbb{Z}a^2 & 0 & 0 & \vdots \\
0 & 0 & 0 & \\
\mathbb{Z}a & 0 & 0 & H^3(K(\mathbb{Z},3)) \otimes H^2(\mathbb{CP}^\infty) \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & H^3(K(\mathbb{Z},3))
\end{array}
$$

for this part of $E_2^{s,t}$, $d_2 = 0$

so $E_3^{s,t} = E_2^{s,t}$

we know $E_\infty^{s,t} = 0$ for $(s,t) \neq (0,0)$ since $PK(\mathbb{Z},3) \simeq *$

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathbb{Z}_{\langle a^3 \rangle} & 0 & 0 & \\
0 & 0 & 0 & \\
\mathbb{Z}_{\langle a^2 \rangle} & 0 & 0 & \vdots \\
0 & 0 & 0 & \\
\mathbb{Z}_{\langle a \rangle} \arrow[ddrrr, "d_3"] & 0 & 0 & H^3(K(\mathbb{Z},3)) \otimes H^2(\mathbb{CP}^\infty) \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & H^3(K(\mathbb{Z},3))
\end{tikzcd}
$$

is an isomorphism

So if we define $s = d_3 a$ then s generates $H^3(K(\mathbb{Z},3))$

so $H^3(K(\mathbb{Z},3)) \cong \mathbb{Z}_{\langle s \rangle}$

and $s \cdot a$ is a generator of $\underset{\displaystyle E_3^{3,2}}{\overset{}{H^3(K(\mathbb{Z},3)) \otimes H^2(\mathbb{CP}^\infty)}}$ ($= E_3^{3,2}$)

$$d_3 a^2 = da \cdot a + a \cdot da$$
$$= s \cdot a + a \cdot s = 2a \cdot s$$

so in $E_3^{s,t}$ we see

$$
\begin{array}{ccccccc}
\langle a^3 \rangle & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & \\
\langle a^2 \rangle & 0 & 0 & \langle a^2 s \rangle & & & \\
0 & 0 & 0 & 0 & & & \\
\langle a \rangle & 0 & 0 & \langle a \cdot s \rangle & & & \\
0 & 0 & 0 & 0 & & & \\
\langle 1 \rangle & 0 & 0 & \langle s \rangle & H^4(K(\mathbb{Z},3)) & H^5(K(\mathbb{Z},3)) & H^6(K(\mathbb{Z},3))
\end{array}
$$

d_3 d_4 d_3 d_4 d_5

note: d_3, d_4 mapping to $E_k^{4,0}$ come from 0

and d_k for $k>4$ also 0 map

so if $E_2^{4,0} \neq 0$ then $E_\infty^{4,0} \neq 0$

$\times$ $PK(\mathbb{Z},3)$ contractible

so $H^4(K(\mathbb{Z},3)) = E_2^{4,0} = 0$

similarly d_3, d_4 into $E_k^{5,0}$ come from 0

but $d_5 : E_5^{0,4} \longrightarrow E_5^{5,0}$

$E_2^{0,4} = E_3^{0,4} = \mathbb{Z}$ gen by a^2

but $d_3 a^2 \neq 0$ so $E_k^{0,4} = 0$ $\quad k>3$

$\therefore$ d_5 into $E_5^{5,0}$ also 0

and $d_k = 0$ for $k>5$ too

so as above $H^5(K(\mathbb{Z},3)) = 0$

now $E_3^{s,t}$ looks like

$$
\begin{tikzcd}[row sep=small, column sep=small]
\langle a^3\rangle & 0 & 0 & \langle a^3 s\rangle & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
\langle a^2\rangle \arrow[rrrdd, "d_3"] & 0 & 0 & \langle a^2 s\rangle & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
\langle a\rangle & 0 & 0 & \langle a\cdot s\rangle \arrow[rrrdd, "d_3"] & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
\langle 1\rangle & 0 & 0 & \langle s\rangle & 0 & 0 & H^6(K(\mathbb{Z},3))
\end{tikzcd}
$$

so

$$
\mathbb{Z}_{\langle a^2\rangle} \xrightarrow{\times 2} \mathbb{Z}_{\langle a,s\rangle} \xrightarrow{d_3} H^6(K(\mathbb{Z},3))
$$

if $\ker d_3$ not $\langle 2a\cdot s\rangle$ then $E_4^{3,2} = \mathbb{Z}/2$ and so is $E_k^{3,2}$ $\forall k$ (no other d_k can kill it but $E_\infty^{3,2} = 0$)

moreover d_3 must be onto $E_3^{6,0}$ or it would live to E_∞, so

$$H^6(K(\mathbb{Z},3)) = E_3^{6,0} = \mathbb{Z}/2$$

recall we want to compute $H_5(K(\mathbb{Z},3))$

the Universal Coefficients Theorem gives

$$0 = H^5(K(\mathbb{Z},3)) \cong \text{free } H_5(K(\mathbb{Z},3)) \oplus \text{tor } H_4(K(\mathbb{Z},3))$$

from above

$\therefore H_5(K(\mathbb{Z},3))$ is all torsion

$$\mathbb{Z}/2 \underset{\uparrow}{=} H^6(K(\mathbb{Z},3)) \cong \text{free } H_6(K(\mathbb{Z},3)) \oplus \text{tor } H_5(K(\mathbb{Z},3))$$

from above

$$\therefore\ H_5(K(\mathbb{Z},3)) \cong \mathbb{Z}/2$$

we are done with the proof but let's compute a bit more of $H^q(K(\mathbb{Z},3))$

we know $E_3^{s,t}$ looks like

$$\begin{array}{ccccccccc}
\langle a^3\rangle & 0 & 0 & \langle a^3 s\rangle & \vdots & \vdots & \mathbb{Z}/2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\langle a^2\rangle & 0 & 0 & \langle a^2 s\rangle & 0 & 0 & \mathbb{Z}/2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\langle a\rangle & 0 & 0 & \langle a\cdot s\rangle & 0 & 0 & \mathbb{Z}/2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\langle 1\rangle & 0 & 0 & \langle s\rangle & 0 & 0 & \mathbb{Z}/2 & H^7
\end{array}$$

$E_3^{7,0}$ must be zero since domains of all d_k mapping to it are shown above so all d_k are 0

but $E_\infty^{7,0} = 0$

$$\therefore\ H^7(K(\mathbb{Z},3)) = E_3^{7,0} = 0$$

so we now know $E_3^{s,t}$ looks like

	0								
	$\langle a^3\rangle$	0	0	$\langle a^3 s\rangle$	$\vdots$	$\vdots$	$\mathbb{Z}/2$	$\vdots$	
	0	0	0	0	0	0	0	0	
4	$\langle a^2\rangle$	0	0	$\langle a^2 s\rangle$	0	0	$\mathbb{Z}/2$	0	
	0	0	0	0	0	0	0	0	
	$\langle a\rangle$	0	0	$\langle a\cdot s\rangle$	0	0	$\mathbb{Z}/2$	0	
	0	0	0	0	0	0	0	0	
	$\langle 1\rangle$	0	0	$\langle s\rangle$	0	0	$\mathbb{Z}/2$	0	H^8
				3					

note: $d_3(a^3) = 3a^2 \cdot s$

$d_3(a^2 s) = 2a\cdot s = 0$ in $\mathbb{Z}/2$

$\therefore E_4^{3,4} = \mathbb{Z}/3$

the only way $E_4^{3,4}$ does not "live to ∞" is if it is onto some $E_k^{s,k}$

only possibility is $E_5^{3,4} \longrightarrow E_5^{8,0}$

also $E_5^{8,0} = E_3^{8,0}$ (since all maps d_k $k=3,4$ in and out of E_k^8 are 0)

$$\therefore d_5 : \underset{\substack{\cong\\ \mathbb{Z}/3}}{E_5^{3,4}} \to \underset{\substack{\cong\\ H^8(K(\mathbb{Z},3))}}{E_5^{8,0}} \text{ is an isomorphism}$$

so we have computed

q	0	1	2	3	4	5	6	7	8
$H^q(K(\mathbb{Z},3))$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$

Thm 14:

$$\boxed{\pi_5(S^3) \cong \mathbb{Z}/2}$$

if we try to compute as we did for $\pi_4(S^3)$ we will run into problems in the spectral sequence because we cannot determine some differentials, so we need a new idea!

given a CW complex we can construct a sequence of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
K(\pi_n(X), n-1) \arrow[r] & X_n \arrow[d] \\
 & X_{n-1} \arrow[d, dotted] \\
 & \vdots \arrow[d] \\
K(\pi_1(X), 0) \arrow[r] & X_1 \arrow[d] \\
 & X
\end{tikzcd}$$

such that 1) X_n is n-connected

ie. $\pi_k(X_n) = 0 \ \forall k \leq n$

2) $\pi_k(X_n) \cong \pi_k(X) \quad \forall k > n$

3) $X_n \to X_{n-1}$ has fiber $K(\pi_n(X), n-1)$

this is called a <u>Whitehead tower</u> of X

<u>note</u>: it generalizes the universal cover and is kind of "dual" to Postnikov towers

lemma 15:

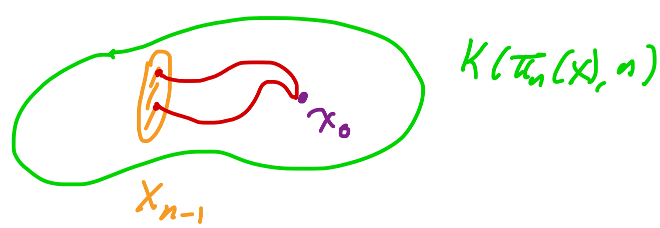
$\boxed{\text{Every CW-complex has a Whitehead tower}}$

Proof: let $X_0 = X$

to construct X_n from X_{n-1} attach cells of dimension $\geq n+2$ to kill π_k for $k \geq n+1$

this gives $K(\pi_n(X), n) = X_{n-1} \cup e^{n+2} \dots$

let Λ_{n-1} = paths in $K(\pi_n(X), n)$ from a fixed base point $x_0 \in K(\pi_n(X), n)$ to X_{n-1}



let $p: \Lambda_{n-1} \to X_{n-1}$ be evaluate at end point

like in the proof of lemma II.5 that

$\Omega(X) \to P(X) \to X$ is a fibration

one can show $p: \Lambda_{n-1} \to X_{n-1}$

is a fibration

recall Ω shifts π_k by one

the fiber is $\Omega(K(\pi_n(X), n)) = K(\pi_n(X), n-1)$

(just take $x_0 \in X_{n-1}$)

consider the exact sequence of fibration

$$\pi_k(K(\pi_n(X), n-1)) \to \pi_k(\Lambda_{n-1}) \to \pi_k(X_{n-1}) \to \pi_{k-1}(K(\pi_n(X), n-1))$$

if $k \geq n+1$, then $\pi_k(\Lambda_{n-1}) \cong \pi_k(X_{n-1}) \cong \pi_k(X)$

induction

if $k \leq n-2$, then $\pi_k(\Lambda_{n-1}) \cong \pi_k(X_{n-1}) = 0$

we are left with

$$0 \to \pi_n(\Lambda_{n-1}) \to \pi_n(X_{n-1}) \xrightarrow{\partial} \pi_{n-1}\big(\overbrace{K(\pi_n(X), n-1)}^{\simeq\, \Omega K(\pi_n(X), n)}\big) \to \pi_{n-1}(\Lambda_{n-1}) \to 0$$

$$\pi_n(X_{n-1}) \cong \pi_n(X), \qquad \pi_{n-1}(K(\pi_n(X), n-1)) \cong \pi_n(X)$$

if ∂ is an isomorphism then $\pi_n(\Lambda_{n-1}) = \pi_{n-1}(\Lambda_{n-1}) = 0$

so if we set $X_n = \Lambda_{n-1}$ then this is the next step in Whitehead tower

for this note

$$X_{n-1} \overset{i}{\hookrightarrow} K(\pi_n(X), n) = X_{n-1} \cup e^{n+2} \cup \dots$$

so i induces an isomorphism on π_n

$$\pi_n(X_{n-1}) \xrightarrow[\cong]{i_*} \pi_n(K(\pi_n(X), n)) \xrightarrow[\cong]{} \pi_{n-1}(\Omega K(\pi_n, n))$$

exercise: exercise show this composition is exactly ∂

Hint: proof $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ uses ∂ map in L.E.S. of fibration

Proof of Thm 14:

Whitehead tower of S^3 is

$$\begin{tikzcd} X_4 \arrow[d] & K(\pi_4(S^3),3) \arrow[l] \\ X_3 \arrow[d] & K(\mathbb{Z},2) \arrow[l] \\ S^3 \end{tikzcd}$$

Hurewicz

$$\pi_5(S^3) \cong \pi_5(X_4) \cong H_5(X_4)$$

we first need to compute the homology of X_3

we have a fibration

$$\begin{tikzcd} X_3 \arrow[d] & K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty \arrow[l] \\ S^3 \end{tikzcd}$$

recall $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[a]$ and

$$H^*(S^3) \cong \begin{cases} \mathbb{Z} & * = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

so Leray-Serre sequence gives

$$H_2^{s,t} = H^s(S^3; H^t(\mathbb{C}P^\infty))$$

so we see

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 \\
\langle a^2 \rangle & 0 & 0 & \langle a^2 u \rangle \\
0 & 0 & 0 & 0 \\
\langle a \rangle & 0 & 0 & \langle a u \rangle \\
0 & 0 & 0 & 0 \\
\langle 1 \rangle & 0 & 0 & \langle u \rangle
\end{array}
$$

$d_2 = 0$ so $E_2 = E_3$

recall $H^k(X_3) = 0$ for $k = 1, 2, 3$
(by Hurewicz)

so E_∞ looks like

$$
\begin{array}{cccc}
0 & & & \\
0 & 0 & & \\
0 & 0 & 0 & \\
\mathbb{Z} & 0 & 0 & 0
\end{array}
$$

H^1 H^2 H^3

$\therefore$ we must have $d_3\, a = u$

thus $d_3\, a^n = n a^{n-1} u$

and we see $E_4 = E_\infty$ is

$$
\begin{array}{cccc}
 & & & \vdots \\
0 & 0 & 0 & \mathbb{Z}/4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}/3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}/2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
$$

Since there is only one nontrivial term on each diagonal $\bigoplus_{s+t=q} E_\infty^{s,t} = H^q(X_3)$

lemma 1

thus we have

by Universal coefficients Th$^{\underline{m}}$

q	0	1	2	3	4	5	6	7	8	9
$H^q(X_3)$	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/4$
$H_q(X_3)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/4$	0

now for the homology of X_4

we apply the homology Leray-Serre sequence to

$$\begin{array}{ccc} K(\pi_4(S^3),3) & \longrightarrow & X_4 \\ & & \downarrow \\ & & X_3 \end{array}$$

the E^2 term is $E^2_{s,t} = H_s(X_3; H_t(K(\pi_4(S^3),3)))$

we computed $H_s(X_3)$ above and

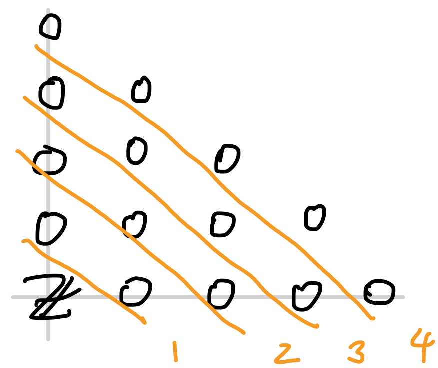
$$H_t(K(\pi_4(S^3),3)) \cong \begin{cases} \mathbb{Z} & t=0 \\ 0 & t=1,2, \\ \pi_4(S^3) & t=3 \end{cases}$$ by Hurewicz

E^2

$$\begin{array}{c|ccccccc} \pi_4(S^3) & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/3 & \end{array}$$

we know $H_k(X_4) = 0$ for $k = 1, 2, 3, 4$

so E^∞ must be



so $E^2_{4,0} = \mathbb{Z}/2$ must die at some point

only possibility is

$$\begin{array}{ccc} d^4 : E^4_{4,0} & \longrightarrow & E^4_{0,3} \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z}_2 & & \pi_4(S^3) \end{array}$$

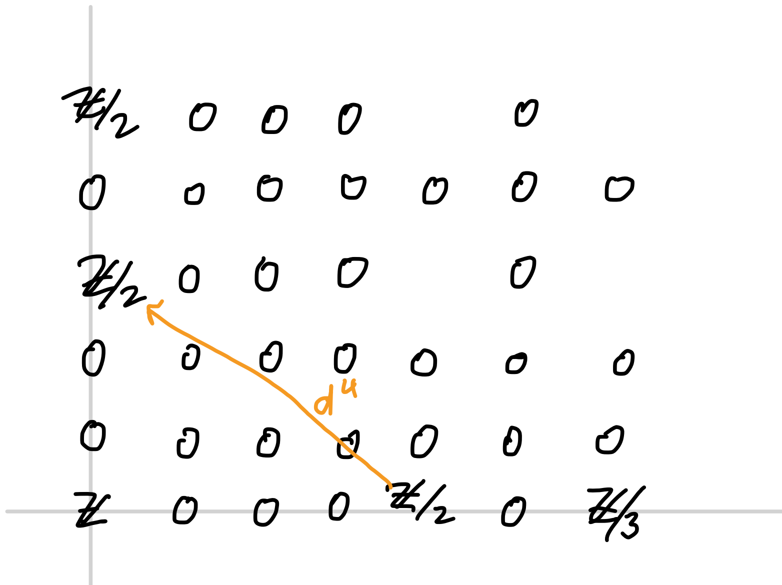
this must be an isomorphism or something lives to E^∞

<u>note</u>: this is another proof $\pi_4(S^3) \cong \mathbb{Z}/2$!

<u>lemma 16</u>:

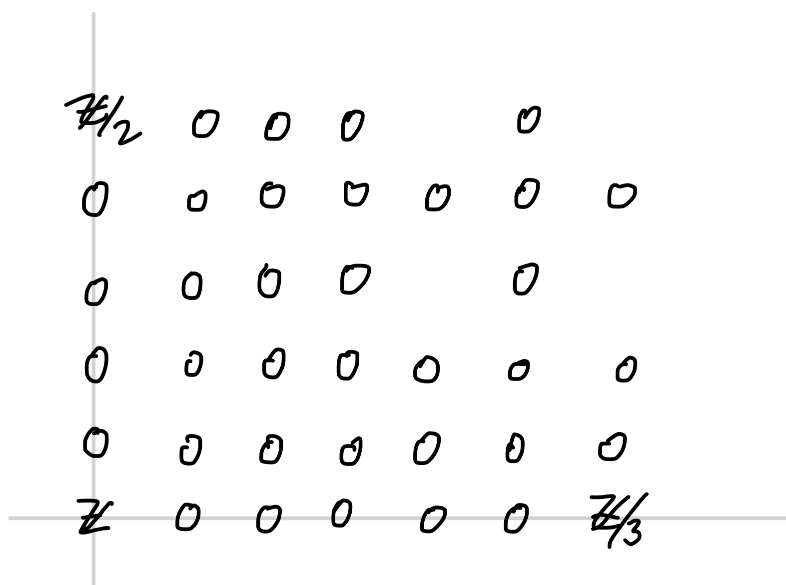
$$\boxed{\begin{array}{l} H_4(K(\mathbb{Z}/2, 3)) = 0 \\ H_5(K(\mathbb{Z}/2, 3)) = \mathbb{Z}/2 \end{array}}$$

so now $E^2_{s,t}$ looks like



$E^2 = E^3$
and d^4 must be an isomorphism

so $E^5_{s,t}$ is

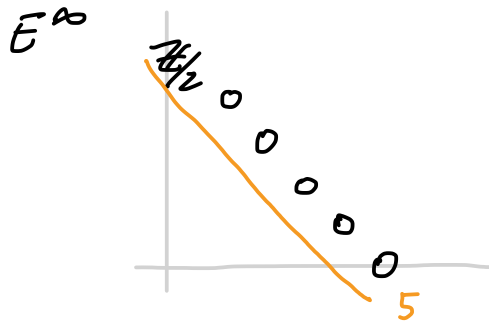


the only possible nonzero differential that hits $E^k_{0,5}$ is

$$d_6 : \underset{\substack{\cong \\ \mathbb{Z}/3}}{E^6_{6,0}} \to \underset{\substack{\cong \\ \mathbb{Z}/2}}{E^6_{0,5}}$$

but there is no nontrivial map $\mathbb{Z}/3$ to $\mathbb{Z}/2$

$\therefore\ E^\infty_{0,5} \cong \mathbb{Z}/2$ and



$\therefore$ lemma 1 says $H_5(X_4) \cong \mathbb{Z}/2$

and we saw earlier $\pi_5(S_3) \cong H_5(X_4)$

we are left to prove lemma

but first we need to compute homology of

$K(\mathbb{Z}/2, 1)$ and $K(\mathbb{Z}/2, 2)$

since $\pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}/2$ and $\begin{array}{c} S^\infty \simeq pt \\ \downarrow \\ \mathbb{RP}^\infty \end{array}$ is its universal

cover we know $\pi_k(\mathbb{RP}^\infty) = 0$ for $k > 1$ by Thm I.18

so $\mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$

exercise: show $H^*(\mathbb{RP}^\infty) = \mathbb{Z}[\alpha]/\langle 2\alpha \rangle$ $\deg \alpha = 2$

$$= \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

Hint: can't use $\begin{array}{c} S^0 \to S^\infty \\ \quad\ \downarrow \\ \quad\ \mathbb{RP}^\infty \end{array}$ since $\mathbb{RP}^\infty$ not simply con.

but
$$\begin{tikzcd} S^1 \arrow[r] & S^\infty \arrow[r] \arrow[d] & \mathbb{RP}^\infty \arrow[dl] & S^1 \arrow[l] \\ & \mathbb{CP}^\infty & & \end{tikzcd}$$

use this

$H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[\alpha]$ $\deg \alpha = 2$

from earlier exercise

also know $H_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$

$\therefore$ $H^2(\mathbb{R}P^\infty) \cong \mathbb{Z}/2 \oplus$ free part   by Universal Coeff Thm

you will see this is 0

lemma 17:

q	0	1	2	3	4	5	6
$H^q(K(\mathbb{Z}/2,2))$	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_q(K(\mathbb{Z}/2,2))$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

Proof: we use the path space fibration (lemma II. 5)

$$\begin{array}{ccc} \Omega K(\mathbb{Z}/2,2) & \longrightarrow & PK(\mathbb{Z}/2,2) \simeq pt \\ \text{\rotatebox{90}{$\simeq$}} & & \downarrow \\ K(\mathbb{Z}/2,1) & & K(\mathbb{Z}/2,2) \end{array}$$

$$H_t(K(\mathbb{Z}/2,2)) = \begin{cases} \mathbb{Z} & t=0 \\ 0 & t=1 \\ \mathbb{Z}/2 & t=2 \end{cases}$$ Hurewicz

so

$$H^t(K(\mathbb{Z}/2,2)) = \begin{cases} \mathbb{Z} & t=0 \\ 0 & t=1,2 \end{cases}$$ Universal Coefficients Thm

so cohomology Leray-Serre gives

$$E_2^{s,t} \quad \begin{array}{|lll} \mathbb{Z}/2\langle\alpha^3\rangle & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{Z}/2\langle\alpha^2\rangle & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{Z}/2\langle\alpha\rangle & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{Z} & 0 & 0 \\ \hline \end{array}$$

in this part $d_2 = 0$ so $E_3 = E_2$

we know $E_\infty^{s,t} = \begin{cases} \mathbb{Z} & (s,t)=(0,0) \\ 0 & \text{otherwise} \end{cases}$

so $d_3 : E_3^{0,2} \to E_3^{3,0}$ must be an isomorphism

$\therefore \; H^3(K(\mathbb{Z},2)) = E_3^{3,0} \cong E_3^{0,2} \cong \mathbb{Z}/2$

and is generated by $\beta = d_3 \alpha$

so now $E_3 = E_2$ (in region drawn) is

$\mathbb{Z}/2\langle\alpha^3\rangle$	0	0		0
0	0	0		0
$\mathbb{Z}/2\langle\alpha^2\rangle$	0	0	$\mathbb{Z}/2\langle\alpha^2\beta\rangle$	0
0	0	0	0	0
$\mathbb{Z}/2\langle\alpha\rangle$	0	0	$\mathbb{Z}/2\langle\alpha\beta\rangle$	0
0	0	0	0	0
$\mathbb{Z}$	0	0	$\mathbb{Z}/2\langle\beta\rangle$	0

since no nontrivial d_k hits this must be 0

$d_3 : E_3^{0,4} \to E_3^{3,2}$ and $d_3 \alpha^2 = 2\alpha\cdot\beta$

i.e. $d_3 = 0$ here

but last chance to kill $E^{3,2}$ is at E_3 page
(all other maps to and from $E^{3,2}$ are 0)

so $d_3 : E_3^{3,2} \to E_3^{6,0}$ must be nontrivial

and if not an isomorphism then

$E_3^{6,0}/\mathrm{im}\, d_3$ lives to ∞ so is an isomorphism

$$\therefore\ H^6(K(\mathbb{Z}/_2, 2)) \cong E_3^{6,0} \cong \mathbb{Z}/_2$$

generated by $d(\alpha\beta) = \beta^2$

so now at E_3 we have

$$\begin{array}{c|ccccccc}
\mathbb{Z}_2\langle\alpha^3\rangle & 0 & 0 & & 0 & & \\
0 & 0 & 0 & & 0 & & \\
\mathbb{Z}_2\langle\alpha^2\rangle & 0 & 0 & \mathbb{Z}/2\langle\alpha^2\beta\rangle & 0 & & \\
0 & 0 & 0 & 0 & 0 & & \\
\mathbb{Z}_2\langle\alpha\rangle & 0 & 0 & \mathbb{Z}/2\langle\alpha\beta\rangle & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\mathbb{Z} & 0 & 0 & \mathbb{Z}/2\langle\beta\rangle & 0 & E_3^{5,0} & \mathbb{Z}/2\langle\beta^2\rangle
\end{array}$$

since $d_3 : E_3^{0,4} \to E_3^{3,2}$ is zero

last chance to kill $E^{0,4}$ is at E_5

and as before $d_5 : E_5^{0,4} \to E_5^{5,0}$ must be an isomorphism

(note $E_5^{5,0} = E_2^{5,0}$)

so $H^5(K(\mathbb{Z}/_2, 2)) = E_5^{5,0} \cong E_5^{0,4} \cong \mathbb{Z}/_2$

thus we have

q	0	1	2	3	4	5	6
$H^q(K(\mathbb{Z}/_2, 2))$	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/_2$

and $H_q(K(\mathbb{Z}/_2, 2))$ follows from Universal Coefficients Th$^{\underline{m}}$

Proof of lemma 16:

we have $K(\mathbb{Z}/2,2) \simeq \Omega K(\mathbb{Z}/2,3) \longrightarrow PK(\mathbb{Z}/2,3) \simeq \{*\}$

$$\begin{tikzcd} & PK(\mathbb{Z}/2,3) \arrow[d] \\ & K(\mathbb{Z}/2,3) \end{tikzcd}$$

so the E_∞ of the cohomology Leray-Serre spectral sequence is

$$\begin{array}{llll} \vdots & & & \iddots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \mathbb{Z} & 0 & 0 & \cdots \end{array}$$

and $E_2^{s,t} = H^s(K(\mathbb{Z}/2,3); H^t(\mathbb{Z}/2,2))$

we know $H^t(\mathbb{Z}/2,2)$ from lemma 17

$$H_s(K(\mathbb{Z}/2,3)) \cong \begin{cases} \mathbb{Z} & s=0 \\ 0 & s=1\,2 \\ \mathbb{Z}/2 & s=3 \end{cases}$$ from Hurewicz

so $$H^s(K(\mathbb{Z}/2,3)) \cong \begin{cases} \mathbb{Z} & s=0 \\ 0 & s=1,2,3 \\ \mathbb{Z}/2 + \text{free} & s=4 \end{cases}$$

so we have

$E_2^{s,t}$

$$\begin{tikzcd} \mathbb{Z}/2 & 0 & 0 & 0 & \mathbb{Z}/2 & 0 \\ \mathbb{Z}/2 & 0 & 0 & 0 & \mathbb{Z}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{Z}/2 \arrow[dddrrrr, "d_4"] & 0 & 0 & 0 & \mathbb{Z}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2 & 0 & H^6 \end{tikzcd}$$

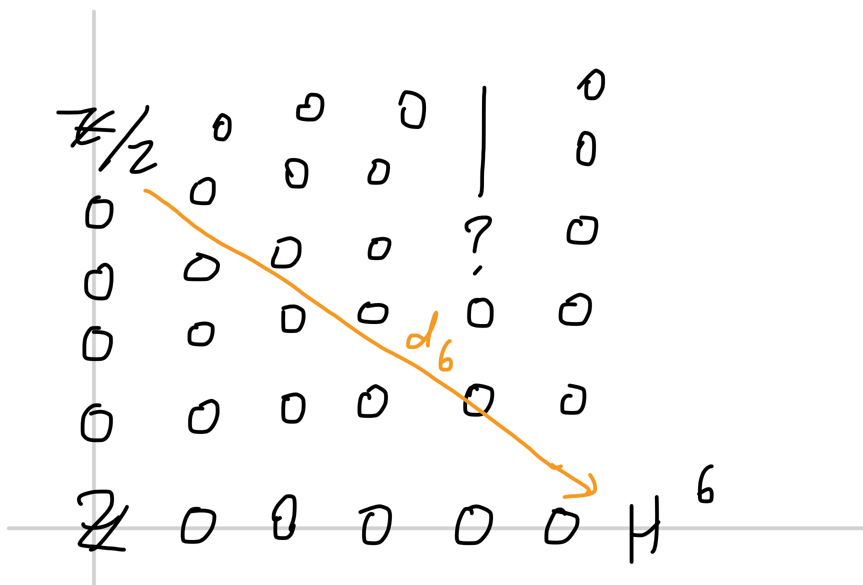
no free part or lives to ∞

must be 0 or lives to ∞

for this part $E_2 = E_3 = E_4$

d_4 must be an isomorphism or something lives to infinity

E_5 and E_6 look like



d_6 must be an isomorphism or something lives to ∞

so we see $H^5 = 0$ and $H^6 = \mathbb{Z}/2$

Universal coefficients says

(1) $\mathbb{Z}/2 = H^6(K(\mathbb{Z}/2,3)) \cong \text{free } H_6(K(\mathbb{Z}/2,3)) \oplus \text{tor}(H_5(K(\mathbb{Z}/2,3))$

(2) $0 = H^5(K(\mathbb{Z}/2,3)) \cong \text{free } H_5(K(\mathbb{Z}/2,3)) \oplus \text{tor}(H_4(\mathbb{Z}/2,3))$

(3) $\mathbb{Z}/2 = H^4(K(\mathbb{Z}/2,3)) \cong \text{free } H_4(K(\mathbb{Z}/2,3)) \oplus \text{tor}(H_3(\mathbb{Z}/2,3))$

(2) $\Rightarrow$ $H_5(K(\mathbb{Z}/2,3))$ is torsion

and (1) says it is $\mathbb{Z}/2$

(3) $\Rightarrow$ $H_4(K(\mathbb{Z}/2,3))$ is torsion

and (2) says it is 0